## 2.6 A numerical method

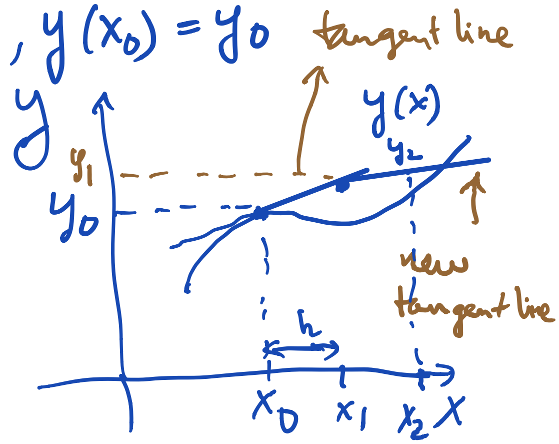
- Analytic solution vs Numerical solution

- Tangent line

Consider $y' = f(x,y)$, $y(x_0) = y_0$

$y'$ = slope of the tangent line

Numerical sol = patching up segments of tangent line




- Euler's method

Linearization : (tangent line)

$$L(x) = \underbrace{f(x_0, y_0)(x - x_0) + y_0}_{\text{eq. of the line through } (x_0, y_0) \text{ with slope } f(x_0, y_0)}$$

Specify step size $h$ :

$x_0$ $\qquad\qquad$ $y_0$

$x_1 = x_0 + h$ $\qquad\qquad$ $\boxed{y_1 = f(x_0, y_0)(x_1 - x_0) + y_0}$

$x_2 = x_1 + h$ $\qquad\qquad$ $y_2 = \underbrace{f(x_1, y_1)}_{\text{new slope at } x_1, y_1}(x_2 - x_1) + y_1$

Ex: $y' = \underline{.2xy}$, $y(1) = 1$, $h = .1$

$x_0 = 1$ $\qquad$ $y_0 = 1$

$x_1 = 1 + .1 = 1.1$ $\qquad$ $y_1 = (.2 \times 1 \times 1)\,h + y_0$

$= (.2)(.1) + 1$

$= 1.02$

$x_2 = x_1 + h = 1.2$ $\qquad$ $y_2 = (.2 \times 1.1 \times 1.02)(.1) + 1.02$

$= 1.0424$

$y' = .2xy$ separable $\rightarrow$ can find the actual solution

Num: $x_1 = 1.1$, $y_1 = 1.02$, $x_2 = 1.2$, $y_2 = 1.0424$

Act: $x_1 = 1.1$, $y_1 = 1.0212$, $x_2 = 1.2$, $y_2 = 1.0450$

Abs error = |True value − Numerical value|

$$\text{Percentage Relative error} = \frac{\text{Abs error}}{|\text{True value}|} \cdot 100$$

## 3.1. Linear Models

- Growth and Decay: Population (biology), economy (economics), Decomposition of chemical materials (physics, chem)

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0, \quad k \text{ constant}$$

- Newton's law of cooling/warming

$$\frac{dT}{dt} = k(T - T_m) \leftarrow \text{first order linear}$$

$T$ : temperature of something
$T_m$ : ambient temperature
$k$ : constant

## Chapter 4.

### 4.1. Theory of Linear Equations

Def : An nth order linear ODE is an eq. of the form :

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x) = g(x) \quad (1)$$

Remark : - Derivatives of $y$ appear isolated from each other

- Each is multiplied by an expression of $x$ (not involving any $y$)

IVP : (1) + set of exactly $n$ initial conditions

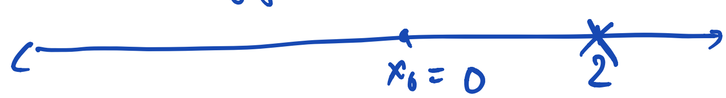
Remark : - General solution expectedly has $n$ different constants.

- $n$ initial conditions $\Rightarrow$ $n$ equations on $n$ constants

- Homogeneous linear ODE: ① but $g(x) \equiv 0$
  Non homogeneous ——— : $g(x) \not\equiv 0$

Thm (Existence & Uniqueness for linear IVPs)
If the function $\{a_j(x)\}_{j=0}^{n}$ and $g(x)$ are continuous on an interval $I = \{a \le x \le b\}$ and $a_n(x) \neq 0$ for all $x \in I$ then there is a unique sol to IVP.

Ex: $(x-2)y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$
$n = 2$, $a_2(x) = x-2$, $a_1(x) = 0$, $a_0(x) = 3$
$g(x) = x$
2nd order linear non homogeneous ODE
- $a_2(x)$, $a_1(x)$, $a_0(x)$, $g(x)$ are continuous on $\mathbb{R}$
- $a_2(x) = x-2 = 0$ iff $x = 2$




Largest interval : $-\infty < x < 2$

Principle of superposition: Non homogeneous vs Homogeneous

$D = \frac{d}{dx}$, $L = a_n(x) D^n + \dots + a_1(x) D + a_0(x)$

Non-hom : $Ly = g(x) \neq 0$ (2)

Homogeneous: $Ly = 0$ (3)

(*) If $y_1, \dots y_n$ are $n$ functions satisfying (3) then $y = c_1 y_1 + \dots + c_n y_n$ is also a solution of (3) for any constants $c_1, \dots, c_n$.

<u>Def</u>: A set of functions $y_1, \dots y_n$ are called linearly independent if

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0 \quad \forall x \in I$$

$\Leftrightarrow$ $c_1 = c_2 \dots = c_n = 0$.

<u>Def</u>: A lin. ind. set of exactly $n$ functions $y_1, \dots y_n$ satisfying (3) is called a fundamental set.

Then the general solution to (3) is given by

$$y = c_1 y_1(x) + \dots + c_n y_n(x)$$

(*) How to determine dependency:

<u>Def</u>: Given a set of functions $y_1, \dots y_k$ we define the Wronskian by

determinant

$$W = W(y_1, \dots y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_k \\ y_1' & y_2' & \dots & y_k' \\ \vdots & & & \\ y_1^{(k-1)} & \dots & \dots & y_k^{(k-1)} \end{vmatrix}$$

Ex: $y_1 = e^{-3x}$, $y_2 = e^{4x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = e^{-3x} 4e^{4x} - (e^{4x}(-3)e^{-3x})$$

$$= 4e^{x} + 3e^{x} = 7e^{x} \neq 0$$

$\Rightarrow \{y_1, y_2\}$ is lin. ind.

Thm: (Wronskian test) The functions $y_1, \dots y_k$ are lin. ind. $\Leftrightarrow W(y_1, \dots y_k) \neq 0$ for $\forall x \in I$.

Thm: There exists a fund set for ③.
$\Leftrightarrow$ There is a general sol of the form
$c_1 y_1(x) + \dots + c_n y_n(x)$

(+) For ② (the non-homogeneous eq)

Thm: Consider $Ly = g(x)$. If $y_p$ is any particular solution and $y_c$ is the complementary solution (solution to $Ly = 0$) then the general sol is $y = y_c + y_p$

Ex: Consider $y'' + y = 1 + x^2$
$y_p = x^2 - 1$, check that $\begin{cases} y_p'' = 2 \\ y_p'' + y = 2 + x^2 - 1 = x^2 + 1 \end{cases}$

$\Rightarrow y_p$ is a particular sol.

The corresponding homogeneous eq:

$$y'' + y = 0$$

$$y_C = c_1 \cos x + c_2 \sin x$$

$y_1 = \cos x$, $y_2 = \sin x$, $\{y_1, y_2\}$ is a fund set

The general sol to the non-hom eq is

$$y = x^2 - 1 + c_1 \cos x + c_2 \sin x$$

Next: - Find solutions to hom. eq.
- Find a particular sol to a non-hom eq.